SMOOTH MANIFOLDS FALL 2022 - HOMEWORK 6

SOLUTIONS

Problem 1. Let G be a connected Lie group. Show that $Z(G) = \ker(\mathrm{Ad})$, where $Z(G) = \{g \in G : gh = hg \text{ for all } h \in G\}$.

Solution. We will show that $Z(G) \subset \ker(\mathrm{Ad})$ and $\ker(\mathrm{Ad}) \subset Z(G)$. First, suppose that $g \in Z(G)$. Then $gh = hg$ for all $h \in G$, and in particular, $L_g = R_g$. Since $\mathrm{Ad}(g) = (L_g)_*$ by definition, it follows that

$$\mathrm{Ad}(g) = (L_g)_* = (R_g)_* = \mathrm{Id}$$

since every vector field in $\mathrm{Lie}(G)$ is right-invariant. Hence $g \in \ker(\mathrm{Ad})$.

Now, assume that $g \in \ker(\mathrm{Ad})$, and let h be in the image of the exponential map, so $h = \exp(X)$ for some $X \in \mathrm{Lie}(G)$. Then

$$ghg{-}1 = g\exp(X)g^{-1} = \exp(\mathrm{Ad}(g)X) = \exp(X) = h,$$

and $gh = hg$. Since G is connected, the image of the exponential map generates G. Hence, g commutes with all elements of G, and $g \in Z(G)$. □

Problem 2. Classify the 2-dimensional connected Lie subgroups of $\mathrm{Heis} = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}$.

Solution. We claim that the only two-dimensional subgroups are of the form

$$G = \left\{ \begin{pmatrix} 1 & ta & s + \frac{ab}{2}t^2 \\ 0 & 1 & tb \\ 0 & 0 & 1 \end{pmatrix} : s, t \in \mathbb{R} \right\}.$$

Indeed, one may check that G is the image of the following Lie subalgebra under exp:

$$(*) \qquad \mathfrak{g} = \left\{ \begin{pmatrix} 0 & ta & s \\ 0 & 0 & tb \\ 0 & 0 & 0 \end{pmatrix} : s, t \in \mathbb{R} \right\}.$$

Since every connected Lie subgroup has a unique associated Lie subalgebra, it suffices to show that the subalgebras $\mathfrak{g}$ are the only 2-dimensional subalgebras of Lie(Heis).

Let $\mathfrak{g} \subset$ Lie(Heis) be an arbitrary two dimensional subalgebra. Choose the following generators of Lie(Heis):

$$X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad Z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then $\mathfrak{g}$ must have two basis vectors which are linearly independent combinations of X, Y and Z, $V_1 = a_1X + b_1Y + c_1Z$ and $V_2 = a_2X + b_2Y + c_2Z$. We claim first that $Z \in \mathfrak{g}$. If V_1 or V_2 is a multiple of Z, we are done. Otherwise, for $i = 1, 2$, $(a_i, b_i) \neq (0, 0)$. Note that Z commutes with X and Y, and $[X, Y] = Z$. Hence

$$[V_1, V_2] = (a_1 b_2 - b_1 a_2) Z.$$

If this is a nonzero vector, we have shown a multiple of Z, and hence Z itself is contained in $\mathfrak{g}$, since it must be closed under Lie brackets. If this is zero, then the vectors (a_1, b_1) and (a_2, b_2) are proportional. But (a_1, b_1, c_1) and (a_2, b_2, c_2) cannot be proportional since V_1 and V_2 are linearly independent. Hence some linear combination of V_1 and V_2 has 0 as the coefficient of X and Y and a nonzero coefficient of Z. That is, Z is in the span of V_1 and V_2. Hence in all cases, $Z \in \mathfrak{g}$.

Finally, observe that any other vector in $V_0 \in \mathrm{Lie(Heis)}$ will span a subalgebra of Lie(Heis), since $Z \in Z(\mathrm{Heis})$. We may without loss of generality assume it is a linear combination of X and Y since we may subtract any multiple of Z and still yield a basis. That is, any two-dimensional subalgebra must be spans of Z and some element $aX + bY$, which are exactly the algebras in (*). □

Problem 3. Let G be the group of transformations of $\mathbb{R}^2$ obtained by compositions of tranlsations and homotheties $x \mapsto \lambda x$ for $\lambda \in \mathbb{R}_+$.

(1) Show that G is center-free (ie, that $Z(G) = \{e\}$).
(2) Find vector fields on $\mathbb{R}^2$ generating the actions by homotheties and translations.
(3) Show that the vector fields of the previous part form the basis of a Lie algebra (ie, that they span a space closed under Lie brackets).
(4) Compute the adjoint representation of its Lie algebra in the basis of the previous part.
(5) Build a matrix group H isomorphic to G.

Solution. Throughout, we let $T_v(x) = x + v$ be the translation along the vector v and $H_\lambda(x) = \lambda x$ be the homothety which is the multiplication by λ.

(1) Suppose that g is a composition of the maps T_{v_i} and H_{λ_j} in some arbitrary, but fixed order. Since the derivative of T_{v_i} is always the identity at every point and the derivative of H_{λ_j} is always $\lambda_j \mathrm{Id}$ at every point, regardless of the order of composition, the derivative of g is some $(\prod \lambda_j) \cdot \mathrm{Id} = \mu_g \cdot \mathrm{Id}$. If $\mu_g = 1$, then g itself is a translation (by the fundamental theorem of calculus), so it does not commute with all homotheties. If $\mu_g \neq 1$, then g does not commute with translations, since if it did,

$$g(x) + v = T_v(g(x)) = g(T_v(x)) = g(x + v) = g(x) + \mu_g v$$

where the last equation follos from the fact that $dg(x) = \mu_g \cdot \mathrm{Id}$ at every point and the fundamental theorem of calculus. In particular, this can only hold if $\mu_g = 1$. Hence, any element of G has something which fails to commute with it.

(2) Observe that the homotheties are a 1-parameter subgroup, and act via the flow $\varphi_t(x) = e^t x$. The generating vector field of this flow is obtained by differentiating in t, yielding

$$X = \frac{\partial}{\partial x_1} x_1 + \frac{\partial}{\partial x_2} x_2$$

Similarly, the translations are generated by the vector fields $Y_i = \frac{\partial}{\partial x_i}$, $i = 1, 2$, since they are generated by the horizontal and vertical flows $\psi_t^{(i)}(x) = x + t e_i$.

(3) We compute the Lie brackets directly:

$$[X, Y_1] = -Y_1 \qquad [X, Y_2] = -Y_2 \qquad [Y_1, Y_2] = 0.$$

Since the pairwise Lie brackets are contained in the span of $\{X, Y_1, Y_2\}$, it follows that their span is a Lie algebra of vector fields.

(4) We compute the adjoint action of $W = tX + v_1Y_1 + v_2Y_2$ in the ordered basis $\{Y_1, Y_2, X\}$:

$$\mathrm{ad}(W) = \begin{pmatrix} -t & 0 & v_1 \\ 0 & -t & v_2 \\ 0 & 0 & 0 \end{pmatrix}$$

(5) Since G is center free, it follows that Ad is an isomorphism onto its image by the first part. Since the image of Ad is generated by elements of the form $\exp(\mathrm{ad}(W))$, $W \in \mathrm{Lie}(G)$, we compute a matrix group isomorphic to G by exponentiating the matrices $\mathrm{ad}(W)$:

$$\mathrm{Ad}(\exp(W)) = \exp(\mathrm{ad}(W)) = \begin{pmatrix} e^{-t} & 0 & (1-e^{-t})/t \cdot v_1 \\ 0 & e^{-t} & (1-e^{-t})/t \cdot v_2 \\ 0 & 0 & 1 \end{pmatrix}$$

By varying $(t, v_1, v_2) \in \mathbb{R}^3$, we see that we may achieve any matrix in the following group, which is hence isomorphic to G:

$$\left\{ \begin{pmatrix} \lambda & 0 & v_1 \\ 0 & \lambda & v_2 \\ 0 & 0 & 1 \end{pmatrix} : \lambda > 0, (v_1, v_2) \in \mathbb{R}^2 \right\}$$

$\square$